TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 159, September 1971

# INTERPOLATION THEOREMS FOR THE PAIRS OF SPACES $(L^p, L^\infty)$ AND $(L^1, L^q)$(1)

BY
GEORGE G. LORENTZ AND TETSUYA SHIMOGAKI

**Abstract.** A Banach space $Z$ has the interpolation property with respect to the pair $(X, Y)$ if each $T$, which is a bounded linear operator from $X$ to $X$ and from $Y$ to $Y$, can be extended to a bounded linear operator from $Z$ to $Z$. If $X=L^p$, $Y=L^\infty$ we give a necessary and sufficient condition for a Banach function space $Z$ on $(0, l)$, $0<l\leqq+\infty$, to have this property. The condition is that $g\prec^p f$ and $f\in Z$ should imply $g\in Z$; here $g\prec^p f$ means that $g^{*p}\prec f^{*p}$ in the Hardy-Littlewood-Pólya sense, while $h^*$ denotes the decreasing rearrangement of the function $|h|$.

If the norms $\|T\|_X$, $\|T\|_Y$ are given, we can estimate $\|T\|_Z$. However, there is a gap between the necessary and the sufficient conditions, consisting of an unknown factor not exceeding $\lambda_p$, $\lambda_p\leqq 2^{1/q}$, $1/p+1/q=1$.

Similar results hold if $X=L^1$, $Y=L^q$. For all these theorems, the complete continuity of $T$ on $Z$ is assured if $T$ has this property on $X$ or on $Y$, and if $Z$ satisfies a certain additional necessary and sufficient condition, expressed in terms of $\|\sigma_a\|_Z$, $a>0$, where $\sigma_a$ is the compression operator $\sigma_a f(t)=f(at)$, $0\leqq t<l$.

1. **Introduction.** Let $X$, $Y$ and $Z$ be Banach spaces, and let $\mathscr{B}(X)$ denote the totality of bounded linear operators acting on $X$, let $\mathscr{B}(X, Y)=\mathscr{B}(X)\cap\mathscr{B}(Y)$. Also, let $\mathscr{B}(X, Y; K_1, K_2)$ denote the set of all operators in $\mathscr{B}(X, Y)$ satisfying $\|T\|_X\leqq K_1$ and $\|T\|_Y\leqq K_2$. The space $Z$ is said to have the *interpolation property for the pair* $(X, Y)$, if for every $T\in\mathscr{B}(X, Y)$, $T$ (or its unique extension $\hat{T}$ to $Z$) belongs to $\mathscr{B}(Z)$. The space $Z$ has the *interpolation property for the pair* $(X, Y)$ *in the strong sense*, if $T$ has the interpolation property for $(X, Y)$ and if $\|T\|_Z$ (or $\|\hat{T}\|_Z$) is majorized by a positive constant depending only on $\|T\|_X$ and $\|T\|_Y$. In the sequel, $I=(0, l)$ will be a (finite or infinite) interval of the real line, and $(X, \|\cdot\|_X)$ will be a Banach function space of locally Lebesgue integrable functions on $I$ satisfying the following conditions:

(1.1) $|g|\leqq|f|, f\in X$ *implies* $g\in X$ *and* $\|g\|_X\leqq\|f\|_X$;

(1.2) *The norm* $\|\cdot\|_X$ *is semicontinuous*:

$$0\leqq f_n\uparrow f,\ \alpha=\sup_{n\geqq 1}\|f_n\|_X<\infty \textit{ imply } f=\bigcup_{n=1}^{\infty} f_n\in X \textit{ and } \|f\|_X=\alpha.$$

Received by the editors July 13, 1970 and, in revised form, September 21, 1970.

*AMS* 1970 *subject classifications.* Primary 46A30.

*Key words and phrases.* Interpolation property, interpolation theorem, quasi-order, rearrangement invariant Banach function space, space monotone with respect to a quasi-order, completely continuous operator, compression operator, Orlicz space.

(1) This work has been supported, in part, by grant no. GP-23566 of the National Science Foundation.

For a positive measurable function $f$, $d_f(y)=m[t : f(t)>y]$, $y\geqq 0$, is the *distribution function* of $f$. Two positive functions $f$, $g$ are *equimeasurable*, $f\sim g$, if they have the same distribution function. The space $X$ is called *weakly rearrangement invariant* (*rearrangement invariant*), if $0\leqq f\in X$ and $f\sim g$ imply $g\in X$ (resp. $\|g\|_X\leqq\gamma\|f\|_X$, where $\gamma$ is a fixed constant independent upon $f$ and $g$). We write $L^p$ for $L^p(I)$, $1\leqq p\leqq\infty$, and $\|\cdot\|_p$ for the $L^p$-norm on $I$. In his paper [2] A. P. Calderón showed that $X$ has the interpolation property for the pair $(L^1, L^\infty)$ if and only if $X$ is rearrangement invariant. In §3 and §4 we shall study the interpolation property for the pairs $(L^p, L^\infty)$, $1\leqq p<\infty$, and $(L^1, L^q)$, $1<q<\infty$, respectively. We characterize the Banach function spaces having the interpolation property for these pairs (Theorems 2 and 3), extending the results of [2], [11]. In §5 the complete continuity of operators acting on interpolated spaces will be dealt with. Results similar to those of [14] will be obtained, and a special case when $X$ is an Orlicz space will be discussed in the last section.

Let $X$ and $Y$ be Banach function spaces consisting of locally integrable functions. By $X+Y$ we denote the set of all functions $f$ of the form $f=f_1+f_2$, where $f_1\in X$ and $f_2\in Y$. If $Z\subset X+Y$, then each operator $T\in\mathscr{B}(X, Y)$ has a natural extension onto $Z$. For $f\in Z$, we write $f=f_1+f_2$, and define $Tf=Tf_1+Tf_2$. Since $T$ is linear, the value of $Tf$ does not depend on the choice of $f_1$ and $f_2$. An extension of $T$ in this sense will be again denoted by $T$.

2. **Quasi-orders.** For a measurable function $f$ on $I(0, l)$, $f^*$ will denote the decreasing rearrangement of $|f|$, that is, the inverse function of $d_{|f|}(y)$, whenever it is finite. By $S$ we denote the set of all positive simple functions, vanishing outside of a set of finite measure. It is easy to see that $f^*$ is defined if $f$ is locally integrable.

The main tool of this paper is different quasi-order relations between measurable functions $f$, $g$. One of them is the Hardy-Littlewood-Pólya relation $g\prec f$ for locally integrable $f$, $g$, which means that

$$\int_0^x g^*(t)\,dt \leqq \int_0^x f^*(t)\,dt, \qquad x\geqq 0. \tag{2.1}$$

Although this relation is classical, some new properties of it were found in [10]. Here is a further property:

THEOREM 1. *Let $g_1+g_2\prec f$, all these functions being locally integrable and positive. Then there exist positive $f_1, f_2$ for which $f=f_1+f_2$, $g_i\prec f_i$, $i=1, 2$.*

LEMMA 1. *Let $g\prec f$, where $g, f$ are positive and $g$ a decreasing function in $S$:*

$$g=\sum_{\nu=1}^n \alpha_\nu\chi_{(c_{\nu-1},\, c_\nu)}, \qquad 0=c_0<\cdots<c_n\leqq l, \quad \alpha_1\geqq\cdots\geqq\alpha_n\geqq 0.$$

*Then there exist mutually disjoint sets $e_\nu$, $\nu=1,\ldots, n$, with the following properties:*

$$me_\nu = c_\nu-c_{\nu-1}, \tag{2.2}$$

$$\alpha_\nu me_\nu \leqq \int f\chi_{e_\nu}\,dt. \tag{2.3}$$

**Proof.** First we assume that $f$ is decreasing. In this case, we shall also have the following:

(2.4) *Each set $e_\nu$ is a finite union of intervals.*

For $n=1$ the assertion holds trivially. Suppose that it holds for $n=k$. Let $n=k+1$. Putting

$$a = \sup\left\{c\colon \int_c^{c+c_1} f\,dt \geqq \alpha_1 c_1,\ c \leqq c_n - c_1\right\},$$

we have $\int_a^{a+c_1} f\,dt=\alpha_1 c_1$, unless $a=c_n-c_1$. Let $\tau_c h$ denote the translation operator, defined by

$$\begin{aligned} \tau_c h(t) &= h(t+c) \quad \text{if } t+c \in I,\\ &= 0 \qquad\quad\ \ \text{otherwise.}\end{aligned}$$

We put

$$\begin{aligned} f_1 &= (f\chi_{(0,a)} + f\chi_{(a+c_1,l)})^* = f\chi_{(0,a)} + \tau_{c_1}(f\chi_{(a+c_1,l)}),\\ g_1 &= \tau_{c_1}\left(\sum_{\nu=2}^{n} \alpha_\nu \chi_{(c_{\nu-1},c_\nu)}\right) = \sum_{\nu=2}^{n} \alpha_\nu \chi_{(c_{\nu-1}-c_1,c_\nu-c_1)}.\end{aligned}$$

We can exclude the possibility that $a=c_n-c_1$, for then $g_1(t)\leqq f_1(t)$ for all $t$. Since

$$\int_0^x g_1\,dt \leqq \int_0^x g\,dt \leqq \int_0^x f\,dt = \int_0^x f_1\,dt \quad \text{if } 0 < x \leqq a,$$

$$\begin{aligned}\int_0^x g_1\,dt &= \int_0^{c_1+x} g\,dt - \alpha_1 c_1 \leqq \int_a^{c_1+x} f\,dt - \int_a^{a+c_1} f\,dt\\ &= \int_0^x f_1\,dt \quad \text{if } a < x \leqq l,\end{aligned}$$

we see that $g_1 \prec f_1$. By the assumption, there exist mutually disjoint sets $\tilde{e}_\nu$, $2\leqq\nu\leqq k+1$, such that (2.2)–(2.4) hold for $f_1$ and $g_1$. Setting $e_1=(a, a+c_1)$ and $e_\nu=\{\tilde{e}_\nu \cap (0,a)\} \cup \{t : t-c_1 \in \tilde{e}_\nu \cap (a,l)\}$, $2\leqq\nu\leqq k+1$, we obtain mutually disjoint sets $e_\nu$, $1\leqq\nu\leqq k+1$, for which all the required conditions hold for $f$ and $g$.

If $f$ is positive but not decreasing, then, since $g\prec f^*$, we can find mutually disjoint measurable sets $e_\nu$, $1\leqq\nu\leqq n$, such that (2.2)–(2.4) hold for $g$ and $f^*$. As each $e_\nu$ is a finite sum of intervals, we can easily find mutually disjoint sets $\tilde{e}_\nu$, $1\leqq\nu\leqq n$, such that $m\tilde{e}_\nu=me_\nu$ and $\int f\chi_{\tilde{e}_\nu}\,dt=\alpha_\nu me_\nu$. Measurable sets $\tilde{e}_\nu$, $1\leqq\nu\leqq n$, thus obtained, satisfy the requirements of Lemma 1.

We can now prove Theorem 1 when $g_1$ and $g_2$, and consequently $g=g_1+g_2$ belong to $S$. Let $e_\nu$, $\nu=1,\dots,n$, be sets of constancy of each of the three functions, with $g_1=\alpha_{\nu1}$, $g_2=\alpha_{\nu2}$ on $e_\nu$. By means of the decreasing rearrangement of $g$ and Lemma 1, we find disjoint sets $\tilde{e}_\nu$ with $m\tilde{e}_\nu=me_\nu$, $\int_{\tilde{e}_\nu} f\,dt \geqq (\alpha_{\nu1}+\alpha_{\nu2})me_\nu$. Then it is possible to decompose each $\tilde{e}_\nu$ into disjoint $\tilde{e}_{\nu1}$, $\tilde{e}_{\nu2}$ such that $\int_{\tilde{e}_{\nu i}} f\,dt \geqq \alpha_{\nu i}me_\nu$, $i=1,2$. We shall have $f\chi_{\tilde{e}_{\nu i}} \succ g_i\chi_{e_\nu}$, $i=1,2$, $\nu=1,\dots,n$. Adding these relations, we obtain $g_i \prec f_i = \sum_{\nu=1}^n f\chi_{\tilde{e}_{\nu i}}$, $f_1+f_2\leqq f$. It is now sufficient to replace $f_2$ by $f-f_1$ to obtain the result.

If $g_1$, $g_2$ are arbitrary positive functions, one finds increasing sequences $g_{1n}\uparrow g_1$, $g_{2n}\uparrow g_2$ from $S$. For the corresponding $f_{1n}, f_{2n}$ one can use weak *-compactness on each set $A$ where $f$ is bounded, and the absolute continuity of the integrals $\int_e f\,dt$ to complete the proof.

REMARK. It is not difficult to show that the functions $f_1, f_2$ of Theorem 1 can be always assumed to be orthogonal (that is, with disjoint supports). However, one cannot, in general, assume that they are decreasing, even if $g_1$, $g_2$ and $f$ are decreasing step functions with just one step.

In [10] another quasi-order $g \prec\!\!\!\cdot\, f$ has been used. With respect to two Banach function spaces this relation means the following. One must have $g, f\in X_1+X_2$, and for each decomposition $f=f_1+f_2$, $f_i\in X_i$, $i=1,2$ of $f$ there should exist a decomposition $g=g_1+g_2$ of $g$, $g_i\in X_i$, $i=1,2$, with the property that $\|g_i\|_{X_i}\leqq \|f_i\|_{X_i}$, $i=1,2$. We are interested here in the case $X_1=L^p$, $X_2=L^\infty$. Then it is easy to see (compare also [10, p. 38]) that $g\prec\!\!\!\cdot\, f$ holds if and only if

$$\|(g^*-y)_+\|_p \leqq \|(f^*-y)_+\|_p,\quad y\geqq 0. \tag{2.5}$$

The quasi-order used in this paper, $g\prec^p f$, where $p\geqq 1$, is defined, for two locally $p$-integrable functions, by the inequality

$$\int_0^x g^{*p}\,dt \leqq \int_0^x f^{*p}\,dt,\qquad x\geqq 0, \tag{2.6}$$

that is, by $g^{*p}\prec f^{*p}$. If one writes (2.6) as $\|g^*\chi_{(0,x)}\|_p\leqq \|f^*\chi_{(0,x)}\|_p$, there is an obvious similarity to (2.5).

From the definition we see that

$$g\prec^p f \text{ is equivalent to } g^*\prec^p f^*. \tag{2.7}$$

By a theorem of Hardy, Littlewood and Pólya, [3], $g\prec^p f$ implies $\Phi(|g|)\prec\Phi(|f|)$, where $\Phi(u)$, $u\geqq 0$, is convex and increasing. In particular,

$$g\prec f \text{ implies } g\prec^p f. \tag{2.8}$$

We also have

$$g_i\prec^p f,\ i=1,2,\ a_1,\alpha_2\geqq 0,\ \alpha_1+\alpha_2=1 \text{ imply } \alpha_1 g_1+\alpha_2 g_2\prec^p f. \tag{2.9}$$

In fact, for $x\in I$ we have, because of the inequality $(f_1+f_2)^*\prec f_1^*+f_2^*$ and (2.8),

$$\int_0^x (\alpha_1 g_1+\alpha_2 g_2)^{*p}\,dt \leqq \int_0^x (\alpha_1 g_1^*+\alpha_2 g_2^*)^p\,dt$$
$$\leqq \alpha_1\int_0^x g_1^{*p}\,dt+\alpha_2\int_0^x g_2^{*p}\,dt \leqq \int_0^x f^{*p}\,dt.$$

LEMMA 2. (i) *Relation $g\prec^p f$ implies $g\prec f$*; (ii) *for each $p>1$, there is a smallest constant $\lambda_p$, $1<\lambda_p\leqq 2^{1/q}$ $(1/p+1/q=1)$, for which $g\prec f$ implies $g\prec^p \lambda_p f$.*

**Proof.** (i) For a given $y \geqq 0$, we consider the function $\Phi(u)=(u^{1/p}-y)_+^p$, which is increasing and convex. Thus, by the theorem of Hardy, Littlewood and Pólya mentioned above $g \prec^p f$ implies $\Phi(g^{*p}) \prec \Phi(f^{*p})$; relation (2.5) follows from this.

(ii) Assume $g \prec f$. If $e_0 \subset I$ is a given set, with $me_0=a>0$, let $\alpha=f^*(a)$, and let $f_2=f^{(\alpha)} \in L^\infty$ be the $\alpha$-truncation of $f$, let $f_1=f-f^{(\alpha)} \in L^p$. There exist $g_i$, $i=1, 2$, with $g=g_1+g_2$, $\|g_1\|_p \leqq \|f-f^{(\alpha)}\|_p$, $\|g_2\|_\infty \leqq \alpha$. Let $e \subset I$, $me \leqq a$. Then

$$\|g\chi_e\|_p \leqq \|g_1\chi_e\|_p+\|g_2\chi_e\|_p \leqq \|g_1\|_p+\alpha a^{1/p} \leqq \|f-f^{(\alpha)}\|_p+\|f^{(\alpha)}\chi_{e_0}\|_p \leqq 2^{1/q}\|f\chi_{e_0}\|_p.$$

We have used here the fact that if $f_1, f_2 \geqq 0$, then $\|f_1\|_p+\|f_2\|_p \leqq 2^{1/q}\|f_1+f_2\|_p$. From (2.10) it follows that $g \prec^p 2^{1/q}f$. We leave to the reader the proof that $\lambda_p>1$.

3. **An interpolation theorem for the pair $L^p$, $L^\infty$.** In this section we assume that $X$ is a Banach function space satisfying $X \subset L^p+L^\infty$ for some $p$, $1 \leqq p < +\infty$. We shall say that $X$ is *monotone with respect to the relation* $\prec^p$, or that $X$ *belongs to the class* $\mathscr{M}^p$ *if* $g \prec^p f$ *and* $f \in X$ *imply* $g \in X$. For $A>0$, we shall say that $X \in \mathscr{M}^p(A)$ *if* $g \prec^p f$ *and* $f \in X$ *imply* $g \in X$ *and* $\|g\|_X \leqq A\|f\|_X$.

LEMMA 3. *If* $X \in \mathscr{M}^p$, *then* $X \in \mathscr{M}^p(A)$ *for some* $A>0$; *moreover*, $X$ *is rearrangement invariant*.

**Proof.** By (2.8) it is clear that $X$ is weakly rearrangement invariant if $X \in \mathscr{M}^p$. Suppose that $\mathscr{M}^p(A)$ is violated for each $A>0$. Then there exist positive functions $f_n, g_n$, $n=1, 2, \ldots$, such that $g_n \prec^p f_n$, $\|g_n\|_X \geqq n$ and $\|f_n\|_X \leqq 2^{-2n}$. Putting $f=\sum_{n=1}^\infty 2^n f_n$, we have $f \in X$ and $2^n g_n \prec^p f$, $n \geqq 1$. By (2.9) we get

$$g = \sum_{n=1}^\infty g_n = \sum_{n=1}^\infty 2^{-n}(2^n g_n) \prec^p f,$$

hence $g \in X$. This, however, contradicts the fact that $\|g\|_X \geqq \|g_n\|_X \geqq n$ for all $n \geqq 1$. Thus, the condition $\mathscr{M}^p(A)$ holds for some $A>0$, and $X$ is necessarily rearrangement invariant.

(For $p=1$, Lemma 3 was given in [12], [16], but the present proof is simpler.)

A space $X \subset L^p$ is *normally imbedded* in $L^p$ if $X$ is dense in $L^p$ and if $\|f\|_p \leqq \|f\|_X$ for all $f \in X$. Each of the Lorentz spaces $\Lambda(C, p)$ [9] (where $C$ is a class of decreasing positive functions $c$ with $\int c\,dt=1$) is normally imbedded in $L^p$ and satisfies $\mathscr{M}^p(1)$. Here is an example in the opposite direction:

EXAMPLE 1. The space $\Lambda_\alpha$, $\alpha=p^{-1}$, $p>1$, with the norm $\|f\|_\Lambda=\alpha\int_0^1 t^{\alpha-1}f^*(t)\,dt$, is normally imbedded in $L^p$. On the other hand, it does not satisfy $\mathscr{M}^p$. Indeed, let $\phi(t)=t^{-\alpha}\log^{-1}(1/t)$, $0<t\leqq e^{-1}$; $=0$, $e^{-1}<t<1$. Then $\phi \in L^p$, but $\|\phi\|_\Lambda=+\infty$. We put, for $0<a<e^{-1}$,

$$\begin{aligned} g_a(t) &= \phi(a), \quad 0 \leqq t \leqq a, & f_a(t) &= \phi(a), \quad 0 \leqq t \leqq b, \\ &= \phi(t), \quad a \leqq t \leqq 1, & &= 0, \quad b \leqq t \leqq 1, \end{aligned}$$

selecting $b$ in such a way that $\|f_a\|_p=\|g_a\|_p$.

Then for each $a$, $g_a \prec^p f_a$, but $\|g_a\|_\Lambda \to \infty$, $\|f_a\|_\Lambda = \|f_a\|_p = \|g_a\|_p \to \|\phi\|_p$ for $a \to 0$. Lemma 3 shows that $\mathcal{M}^p$ is violated.

LEMMA 4. *Assume that $f_0, g_0, g_0 \in S$ are positive and that $g_0 \prec^p f_0$. Then there exists a positive operator $T \in \mathcal{B}(L^p, L^\infty; 1, 1)$ with the property that $g_0 \leqq Tf_0$.*

**Proof.** Let $g_0$ be given by

$$g_0 = \sum_{\nu=1}^{n} a_\nu \chi_{\tilde{e}_\nu}, \qquad \alpha_1 \geqq \cdots \geqq \alpha_n \geqq 0, \ \tilde{e}_\nu \cap \tilde{e}_\mu = \varnothing, \nu \neq \mu. \tag{3.1}$$

By Lemma 1 (applied to $g_0^{*p}$) there exist disjoint subsets $e_\nu$, $\nu = 1, \ldots, n$, of $I$ for which $me_\nu = m\tilde{e}_\nu$ and

$$\int_0^l f_0^p \chi_{e_\nu}\, dt \geqq \alpha_\nu^p m e_\nu, \qquad \nu = 1, \ldots, n. \tag{3.2}$$

We denote by $h_\nu$ positive functions with the properties that $\|h_\nu\|_q = 1$ $(1/p + 1/q = 1)$ and

$$\langle f_0 \chi_{e_\nu}, h_\nu \rangle = \int_0^l f_0 \chi_{e_\nu}(t) h_\nu(t)\, dt = \|f_0 \chi_{e_\nu}\|_p.$$

We define an operator $T$ on the set of all locally $p$-integrable functions by

$$Tf = \sum_{\nu=1}^{n} \frac{\langle f \chi_{e_\nu}, h_\nu \rangle}{\|\chi_{e_\nu}\|_p} \chi_{\tilde{e}_\nu}.$$

Clearly $T$ is positive and linear and

$$\|Tf\|_p^p \leqq \sum_{\nu=1}^{n} \|f\chi_{e_\nu}\|_p^p \leqq \|f\|_p^p$$

for all $f \in L^p$. On the other hand, for any $f \in L^\infty$,

$$|\langle f\chi_{e_\nu}, h_\nu \rangle| \leqq \|f\chi_{e_\nu}\|_p \leqq \|f\|_\infty \|\chi_{e_\nu}\|_p, \qquad 1 \leqq \nu \leqq n.$$

Consequently, $T \in \mathcal{B}(L^p, L^\infty; 1, 1)$. Furthermore by (3.2),

$$Tf_0 = \sum_{\nu=1}^{n} \frac{\|f_0 \chi_{e_\nu}\|_p}{\|\chi_{e_\nu}\|_p} \chi_{\tilde{e}_\nu} \geqq \sum_{\nu=1}^{n} \alpha_\nu \chi_{\tilde{e}_\nu} = g_0.$$

Now we can prove

THEOREM 2. *Let $X$ be a Banach function space over $I$ with $X \subset L^p + L^\infty$. The necessary and sufficient condition for $X$ to have the interpolation property for the pair $(L^p, L^\infty)$ or, equivalently, this property in the strong sense for $(L^p, L^\infty)$ is that $X \in \mathcal{M}^p$.*

**Proof.** First let $X \in \mathcal{M}^p$. By Lemma 3, $X \in \mathcal{M}^p(A)$ for some $A > 0$. Let $g, f \in X$ and $g \prec f$ (with respect to $L^p, L^\infty$). Then by Lemma 2, $g \prec^p \lambda_p f$, and so $\|g\|_X \leqq$

$A\lambda_p\|f\|_X$. Now if $T\in\mathscr{B}(L^p, L^\infty; 1, 1)$, then for each $f\in X$, $Tf\prec f$. It follows that $T$ maps $X$ into itself and that $\|T\|_X\leqq A\lambda_p$. And if $0\neq T\in\mathscr{B}(L^p, L^\infty; K_p, K_\infty)$, then $\alpha T\in\mathscr{B}(L^p, L^\infty; 1, 1)$, where $\alpha^{-1}=\text{Max}\,(K_p, K_\infty)$. This shows that $X$ has the interpolation property in the strong sense.

Conversely, suppose that $X$ has the interpolation property for the pair $(L^p, L^\infty)$, but fails to satisfy $\mathscr{M}^p$. Then there exist positive functions $f$ and $g$ such that $f\in X$, $\|f\|_X=1$, $g\prec^p f$, but $g\notin X$. Let $0\leqq g_n\in S$ and $g_n\uparrow g$. As $g_n\prec^p f$, there exist, by Lemma 4, positive operators $T_n\in\mathscr{B}(L^p, L^\infty; 1, 1)$ such that $g_n\leqq T_n f$ for each $n\geqq 1$. This implies that $g_n\in X$ for each $n\geqq 1$. Since $\|\cdot\|_X$ satisfies (1.2), $\|g_n\|_X\uparrow\infty$ holds. We may therefore assume without loss of generality that $\|g_n\|_X>n\cdot 2^n$, $n\geqq 1$. It follows that $\|T_n\|_X\geqq\|T_n f\|_X>n\cdot 2^n$, $n\geqq 1$. Putting $T=\sum_{n=1}^{\infty}2^{-n}T_n$, we obtain a positive operator belonging to $\mathscr{B}(L^p, L^\infty; 1, 1)$. On the other hand, $\|Tf\|_X\geqq\|2^{-n}T_n f\|_X>n$ holds for each $n$, since $T_n$ is a positive operator. This contradicts the fact that $Tf\in X$, and shows that the condition is necessary.

From the proof above, we have immediately

COROLLARY 1. *If $X$ satisfies the condition $\mathscr{M}^p(A)$ and $T\in\mathscr{B}(L^p, L^\infty; K_p, K_\infty)$, then*

$$Tf\prec^p\lambda_p\,\text{Max}\,(K_p, K_\infty)f \quad \textit{for each } f\in L^p; \tag{3.3}$$

$$\|T\|_X\leqq A\lambda_p\,\text{Max}\,(K_p, K_\infty). \tag{3.4}$$

In the last inequalities, $\lambda_p\leqq 2^{1/q}\leqq 2$. We shall show that $\lambda_p$ cannot be here replaced by 1.

EXAMPLE 2. For each $p$, $1<p<+\infty$, there exists an operator $T\in\mathscr{B}(L^p, L^\infty; 1, 1)$, for which $Tf\prec^p f$ is not true for some $f$.

Let $\alpha>1$ be chosen so that $c=\frac{1}{2}(\alpha^{p-1}+1)^p/(\alpha^p+1)^{p-1}<1$ (actually, this is true for any $\alpha>1$). We define

$$\begin{aligned} f_0(x) &= \alpha \quad \text{on } (0,\tfrac{1}{2}), & g_0(x) &= \beta \quad \text{on } (0, c),\\ &= 1 \quad \text{on } (\tfrac{1}{2}, 1), & &= 0 \quad \text{on } (c, 1), \end{aligned}$$

where $\beta=(\alpha^p+1)/(\alpha^{p-1}+1)$. An easy calculation shows that

$$\|g_0\|_p=\|f_0\|_p, \tag{3.5}$$

$$\|f_0^{p-1}\|_1\|g_0\|_\infty=\|f_0\|_p^p. \tag{3.6}$$

We define the positive operator

$$Tf=\frac{1}{\|f_0\|_p^p}\langle f, f_0^{p-1}\rangle g_0. \tag{3.7}$$

Since $\|h^{p-1}\|_q=\|h\|_p^{p-1}$ for $h\geqq 0$, it follows from (3.5) that, if $f\in L^p$,

$$\|Tf\|_p\leqq\frac{1}{\|f_0\|_p^p}\|f\|_p\|f_0^{p-1}\|_q\|g_0\|_p=\|f\|_p,$$

and from (3.6) that, if $f\in L^\infty$,

$$\|Tf\|_\infty\leqq\frac{1}{\|f_0\|_p^p}\|f\|_\infty\|f_0^{p-1}\|_1\|g_0\|_\infty=\|f\|_\infty.$$

Also, $Tf_0 = g_0$. However, $g_0 \prec^p f_0$ is incorrect, since

$$\int_0^c g_0^p \, dt = \|f_0\|_p^p > \int_0^c f_0^p \, dt. \tag{3.8}$$

EXAMPLE 3. There exists a space $X \in \mathcal{M}^p(1)$, and an operator $T \in \mathcal{B}(L^p, L^\infty; 1, 1)$, for which $\|T\|_X > 1$.

In the notations of the last example, we take $X = L^p$ with the norm $\|f\|_X = \|f^* \chi_{(0,c)}\|_p$. It is immediately clear that $g \prec^p f$ implies $\|g\|_X \leqq \|f\|_X$, so that $X \in \mathcal{M}^p(1)$. For the operator (3.7) we have $g_0 = Tf_0$, but $\|g_0\|_X > \|f_0\|_X > 0$ by (3.8).

If $g = Tf$ and $T \in \mathcal{B}(L^p, L^\infty; 1, 1)$, then we have $g \prec f$. We shall show that the converse is not true, in general. This will also show that one cannot replace the relation $\prec^p$ by $\prec$ in Lemma 4.

EXAMPLE 4. Let $p > 1$ be an integer, and let $f_0$ and $g_0$ be the functions of the Example 2. We put $f_1 = f_0 + \mathbf{1}$ and $g_1 = g_0 + \mathbf{1}$, where $\mathbf{1}$ denotes the characteristic function of (0, 1). Let

$$G(t) = \|f_0 + t\mathbf{1}\|_p^p - \|g_0 + t\mathbf{1}\|_p^p, \qquad t \geqq 0.$$

Using (3.5), (3.6) and elementary calculations (for instance, with induction in $k$) we can show that

$$G(0) = G'(0) = 0, \quad G^{(k)}(0) \geqq 0, \qquad 2 \leqq k \leqq p.$$

It follows that $G(t) \geqq 0$ for all $0 \leqq t \leqq 1$, hence we have $g_1 \prec f_1$ on account of (2.5).

Now suppose that there exists an operator $T \in \mathcal{B}(L^p, L^\infty; 1, 1)$ such that $Tf_1 = g_1$. Since $\|T\mathbf{1}\|_\infty \leqq 1$, we have $0 \leqq g_1 - \mathbf{1} \leqq Tf_1 - T\mathbf{1}$, hence

$$\|g_0\|_p = \|g_1 - \mathbf{1}\|_p \leqq \|Tf_1 - T\mathbf{1}\|_p \leqq \|f_1 - \mathbf{1}\|_p = \|f_0\|_p.$$

From (3.5) it follows that $T\mathbf{1} = \mathbf{1}$ and $Tf_0 = g_0$. Since $\chi_{(1/2,1)} = (\alpha\mathbf{1} - f_0)/(\alpha - 1)$, we have

$$T\chi_{(1/2,1)} = (\alpha - 1)^{-1}(\alpha\mathbf{1} - g_0).$$

The last function has values $\alpha/(\alpha-1) > 1$ on the interval $(c, 1)$, hence $\|T\chi_{(1/2,1)}\|_\infty > 1$, a contradiction. Consequently, there does not exist an operator $T \in \mathcal{B}(L^p, L^\infty; 1, 1)$ with the property $Tf_1 = g_1$.

4. **Interpolation theorems for the pair $L^1$, $L^q$.** In this section we assume that $X$ is a Banach function space for which $X \subset L^1 + L^q$, $1 < q < +\infty$, and that $p$ is the conjugate exponent, $1/p + 1/q = 1$. We define a quasi-order relation $\prec_q$. We write $f_1 \prec_q f_2$, $f_1, f_2 \in L^q$, if there exists, for every $g_1 \in L^p$, a $g_2 \in L^p$ such that both $g_2 \prec^p g_1$ and $\langle f_1, g_1 \rangle \leqq \langle f_2, g_2 \rangle$. For example, $0 \leqq f_1 \leqq f_2$ implies $f_1 \prec_q f_2$, for here we can take $g_2 = |g_1|$. We begin with some properties of the relation $\prec_q$. For given $f, g \geqq 0$, there exists a $\tilde{g} \geqq 0$ with the properties $g \sim \tilde{g}$ and $\langle f^*, g \rangle = \langle f, \tilde{g} \rangle$, [8, p. 61]. From this, using (2.7), it is not difficult to derive

$$f_1 \prec_q f_2 \textit{ if and only if } f_1^* \prec_q f_2^*. \tag{4.1}$$

If $f_1 \prec f_2$, then for each $g_1$ we have $\langle f_1^*, g_1\rangle \leqq \langle f_1^*, g_1^*\rangle \leqq \langle f_2^*, g_1^*\rangle$. Hence, by (4.1),

$$f_1 \prec f_2 \textit{ implies } f_1 \prec_q f_2. \tag{4.2}$$

Similar to (2.9) is the property

$$f_i \prec_q f,\ i = 1, 2, \textit{ and } \alpha_1, \alpha_2 \geqq 0,\ \alpha_1+\alpha_2 = 1 \textit{ imply } \alpha_1 f_1+\alpha_2 f_2 \prec_q f. \tag{4.3}$$

In fact, for each $g \in L^p$, we can find $g_1$ and $g_2$ such that $g_i \prec^p g$ and $\langle f_i, g\rangle \leqq \langle f, g_i\rangle$, $i=1, 2$. Hence,

$$\langle \alpha_1 f_1+\alpha_2 f_2, g\rangle \leqq \langle f, \alpha_1 g_1\rangle+\langle f, \alpha_2 g_2\rangle = \langle f, \alpha_1 g_1+\alpha_2 g_2\rangle,$$

where $\alpha_1 g_1+\alpha_2 g_2 \prec^p g$ by (2.9). Since $g$ is arbitrary, we get (4.3).

For a Banach function space $X$, $X'$ will denote the *conjugate space of* $X$, that is, the space of all measurable functions $g$ such that

$$\|g\|_{X'} = \sup\{|\langle f, g\rangle|; f\in X,\ \|f\|_X \leqq 1\} < \infty.$$

For any operator $T$ acting on $X$, $T'$ will denote the *conjugate operator* of $T$ acting on the conjugate space $X'$. Note that $T\in\mathscr{B}(L^1, L^q; K_1, K_q)$ implies $T' \in \mathscr{B}(L^p, L^\infty; K_q, K_1)$.

LEMMA 5. *If* $T\in\mathscr{B}(L^1, L^q; 1, 1)$, *then*

$$Tf \prec_q \lambda_p f \quad \textit{for each } f\in L^q. \tag{4.4}$$

**Proof.** We have $T' \in\mathscr{B}(L^p, L^\infty; 1, 1)$, hence $T'g \prec^p \lambda_p g$ holds for every $g \in L^p$, by (3.3). If $f\in L^q$ and $g_1 \in L^p$ are given, we select $g_2=(1/\lambda_p)T'g_1$. Then $g_2 \prec^p g_1$ and $\langle Tf, g_1\rangle=\langle f, T'g_1\rangle=\langle \lambda_p f, g_2\rangle$, and we have proven (4.4).

We shall use the following monotony conditions for a Banach function space $X$:

$X \in \mathscr{M}_q$, *if* $g \prec_q f, f\in X$ *imply* $g \in X$;

$X \in \mathscr{M}_q(A)$, *if* $g \prec_q f, f\in X$ *imply* $g \in X$ *and* $\|g\|_X \leqq A\|f\|_X$.

With the same proof as for Lemma 3 we have

LEMMA 6. *If* $X \in \mathscr{M}_q$, *then* $X \in \mathscr{M}_q(A)$ *for some* $A>0$; *moreover, $X$ is rearrangement invariant.*

LEMMA 7. *If the space $X$ does not satisfy the condition* $\mathscr{M}_q(A)$, *then there exists a positive operator* $T\in\mathscr{B}(L^1, L^q; 1, 1)$ *and a function* $0\leqq f\in X$ *for which* $\|Tf\|_X > A\|f\|_X$.

**Proof.** We shall first show that under the assumptions of Lemma 7, the conjugate space $X'$ of $X$ does not satisfy $\mathscr{M}^p(A)$. There exist functions $f_1, f_2 \in X$ such that $f_1 \prec_q f_2$ and $\|f_1\|_X > A\|f_2\|_X$. For any $\varepsilon>0$ satisfying $(1-\varepsilon)\|f_1\|_X > A\|f_2\|_X$, we can find, by virtue of the reflexivity of the semicontinuous norm $\|\cdot\|_X$, a function $g_1 \in X' \cap L^p$ such that $\|g_1\|_{X'}=1$ and $(1-\varepsilon)\|f_1\|_X \leqq \langle f_1, g_1\rangle$. Since $f_1 \prec_q f_2$, there exists a function $g_2 \in L^p$ for which $g_2 \prec^p g_1$ and $\langle f_1, g_1\rangle \leqq \langle f_2, g_2\rangle$. This implies

$$A\|f_2\|_X < (1-\varepsilon)\|f_1\|_X \leqq \|f_2\|_X\|g_2\|_{X'}.$$

Thus, we have obtained two functions $g_1, g_2 \in X'$, for which $g_2 \prec^p g_1$, but $\|g_2\|_{X'} > A\|g_1\|_{X'}$, contradicting the condition $\mathcal{M}^p(A)$.

For $g_1$ and $g_2$, obtained above, we may assume $g_1, g_2 \geqq 0$. Since $\|\cdot\|_{X'}$ is also semicontinuous, we can select an $h \in S \cap X'$ such that $0 \leqq h \leqq g_2$ and $\|h\|_{X'} > A\|g_1\|_{X'}$. By Lemma 4 there exists a positive operator $T \in \mathscr{B}(L^p, L^\infty; 1, 1)$ for which $Tg_1 \geqq h$. Choose an $\varepsilon > 0$ such that $(1-\varepsilon)\|h\|_{X'} > A\|g_1\|_{X'}$. There exists a function $0 \leqq \tilde{f} \in X$, $\|\tilde{f}\|_X = 1$ with the property $\langle \tilde{f}, h\rangle \geqq (1-\varepsilon)\|h\|_{X'}$. It follows that $(1-\varepsilon)\|h\|_{X'} \leqq \langle \tilde{f}, h\rangle \leqq \langle \tilde{f}, Tg_1\rangle \leqq \|T'\tilde{f}\|_X \|g_1\|_{X'}$. Consequently, we get $\|T'\tilde{f}\|_X > A$, for the positive operator $T' \in \mathscr{B}(L^1, L^q; 1, 1)$.

Now we can state our interpolation theorem for the pair $(L^1, L^q)$.

THEOREM 3. *Let $X$ be a Banach function space over $I$ with $X \subset L^1 + L^q$. The necessary and sufficient condition for $X$ to have the interpolation property for the pair $(L^1, L^q)$, or, equivalently, this property for $(L^1, L^q)$ in the strong sense, is that $X \in \mathcal{M}_q$.*

**Proof.** First let $X \in \mathcal{M}_q$. By Lemma 6, $X \in \mathcal{M}_q(A)$ for some $A > 0$. Let $0 \neq T \in \mathscr{B}(L^1, L^q; K_1, K_q)$, we put $\alpha^{-1} = \operatorname{Max}(K_1, K_q)$. Then $\alpha T \in \mathscr{B}(L^1, L^q; 1, 1)$ and so $\alpha Tf \prec_q \lambda_p f$ holds for all $f \in L^q$ by Lemma 5. Thus, $f \in L^q \cap X$ implies $Tf \in X$ and $\|Tf\|_X \leqq \lambda_p A\alpha^{-1}\|f\|_X$. We extend this relation to all $f \in X$. Since $f \in L^1 + L^q$, all truncations $f^{(n)}$ belong to $L^q$, and all differences $f - f^{(n)}$ belong to $L^1$ for $n = 1, 2, \ldots$. Since $|f - f^{(n)}| \to 0$ a.e. and $\|T(f-f^{(n)})\|_1 \leqq K_1\|f - f^{(n)}\|_1$, we have $\|Tf - Tf^{(n)}\|_1 \to 0$. Taking, if necessary, a subsequence, we can assume that the sequence $Tf^{(n)}$, $n = 1, 2, \ldots$, converges a.e. to $Tf$. By (1.2) and the semicontinuity of $\|\cdot\|_X$ we have

$$\begin{aligned}\|Tf\|_X &\leqq \liminf_{n\to\infty} \|Tf^{(n)}\|_X \\ &\leqq \liminf_{n\to\infty} \lambda_p A\alpha^{-1}\|f^{(n)}\|_X \leqq \lambda_p A\alpha^{-1}\|f\|_X.\end{aligned}$$

This shows that $T \in \mathscr{B}(X)$, and that $\|T\|_X \leqq \lambda_p A\alpha^{-1}$.

The necessity of the condition $\mathcal{M}_q(A)$ follows exactly as in the proof of Theorem 2.

COROLLARY 2. *If $X \in \mathcal{M}_q(A)$ and $T \in \mathscr{B}(L^1, L^q; K_1, K_q)$, then $T \in \mathscr{B}(X)$ and*

$$\|T\|_X \leqq \lambda_p A \operatorname{Max}(K_1, K_q). \tag{4.5}$$

5. **Complete continuity of operators in interpolation theorems.** In this section we give necessary and sufficient conditions for the space $X$ in order that every operator $T$ in $\mathscr{B}(L^p, L^\infty)$ (or in $\mathscr{B}(L^1, L^q)$) should be completely continuous on $X$ if $T$ is completely continuous on one of the spaces of the pair. We assume that $X \subset L^p + L^\infty$ (or $X \subset L^1 + L^q$) for the pair $(L^p, L^\infty)$ (respectively, $(L^1, L^q)$). The basic idea of the arguments below is due to the paper [4], and the setting and the proofs follow the lines of [11], [14]. The conditions are given in terms of the norms of compression operators. We denote by $\sigma_a$, $a > 0$, the *compression operator*:

$$\begin{aligned}\sigma_a f(t) &= f(at) \quad \text{if } 0 < at < l, \\ &= 0 \qquad\quad \text{otherwise.}\end{aligned} \tag{5.1}$$

For any rearrangement invariant space $(X, \|\cdot\|_X)$ with $\gamma=1$, we have $\sigma_a \in \mathscr{B}(X)$ and (see [13])

$$\|\sigma_a\|_X \leqq 1 \quad \text{if } a \geqq 1; \qquad 1 \leqq \|\sigma_a\|_X \leqq a^{-1} \quad \text{if } 0 < a \leqq 1. \tag{5.2}$$

It is clear that $\sigma_{ab}=\sigma_a\sigma_b$ if $b\geqq 1$, or if $0<a$, $b\leqq 1$. It follows from this and (5.2) that

$$\|\sigma_a\|_X \leqq (c/a)\|\sigma_c\|_X \quad \text{if } 0 < a \leqq c,\ c > 1. \tag{5.3}$$

The norms $\|\sigma_a\|_X$, which play an important role in the theory of function spaces, have been discussed in [1], [13], [14]. We improve the inequality (3.4) of Corollary 1.

LEMMA 8. *If $X$ satisfies the condition $\mathscr{M}^p(A)$, $1\leqq p<\infty$, then, for every $0\neq T\in\mathscr{B}(L^p, L^\infty; K_p, K_\infty)$,*

$$\|T\|_X \leqq A\lambda_p K_\infty \|\sigma_a\|_X, \tag{5.4}$$

*where $a=K_\infty^p\cdot K_p^{-p}$.*

**Proof.** In the assumptions of the lemma, both operators $T'=K_\infty^{-1}T\sigma_{a^{-1}}$ and $T''=K_\infty^{-1}\sigma_{a^{-1}}T$ belong to $\mathscr{B}(L^p, L^\infty; 1, 1)$, as can be easily seen. If $a\geqq 1$, $\sigma_{a^{-1}}\sigma_a=I$ on $X$, hence $T=K_\infty T'\sigma_a$; if $0<a<1$, $\sigma_a\sigma_{a^{-1}}=I$ on $X$, hence $T=K_\infty\sigma_a T''$. In both cases, (5.4) follows from Corollary 1.

An operator $A$ on the set of locally integrable functions is called an *averaging operator* if $A$ is defined by

$$Af = \sum_{\nu=1}^{n} (me_\nu)^{-1}\langle f, \chi_{e_\nu}\rangle\chi_{e_\nu}, \tag{5.5}$$

where $me_\nu<\infty$, $e_\nu\cap e_\mu=\varnothing$, if $\nu\neq\mu$ and $n\geqq 1$. For convenience, we sometimes denote the operator (5.5) by $A_g$, where $g=\sum_{\nu=1}^n \alpha_\nu\chi_{e_\nu}$, is any function in $S$ corresponding to the sets $e_1,\dots,e_n$. It is clear that $A_g g=g$ for all $g\in S$. An averaging operator $A$ belongs to $\mathscr{B}(L^1, L^\infty)$. If $X$ is rearrangement invariant, then, because of the relation $Af\prec f$, $A$ and $I-A$ belong to $\mathscr{B}(X)$. Moreover, $A$ is completely continuous. For each $p$, $1\leqq p<\infty$, there exists a sequence of averaging operators $A_n$, $n=1,2,\dots$, which converges in $L^p$ strongly to the identity operator $I_n$ [5, p. 21].

We have

THEOREM 4. *Let $X$ satisfy $\mathscr{M}^p(A)$, $1\leqq p<\infty$. In order that every operator $T\in\mathscr{B}(L^p, L^\infty)$, which is completely continuous on $L^p$, should be also completely continuous on $X$, it is necessary and sufficient that*

$$\lim_{a\to\infty} \|\sigma_a\|_X = 0. \tag{5.6}$$

**Proof.** Assume that (5.6) is satisfied. The image $TV$ of the unit ball $V$ in $L^p$ has compact closure in $L^p$. We select a sequence $A_n$, $n=1,2,\dots$, of averaging operators converging strongly to $I$ in $L^p$. Then

$$\lim_{n\to\infty}\left\{\sup_{f\in V}\|(I-A_n)Tf\|_p\right\} = 0,$$

hence $\lim_{n\to\infty} \|(I-A_n)T\|_p = 0$. Since $\|(I-A_n)T\|_\infty \leqq \|T\|_\infty$, putting $a_n = (\|(I-A_n)T\|_\infty \|(I-A_n)T\|_p^{-1})^p$ and $c_n = (\|T\|_\infty \|(I-A_n)T\|_p^{-1})^p$, we have $a_n \leqq c_n$ and $c_n \to \infty$. Using (5.4) and (5.3) we obtain

$$\begin{aligned}\|(I-A_n)T\|_X &\leqq A\lambda_p \|(I-A_n)T\|_\infty \|\sigma_{a_n}\|_X \\ &\leqq A\lambda_p \|T\|_\infty \|\sigma_{c_n}\|_X \to 0.\end{aligned}$$

Since $T$ is the uniform limit of the operators $A_nT$, $n=1, 2, \ldots$, which are completely continuous on $X$, $T$ also has the property.

Conversely, assume that (5.6) is not valid for $X$. It has been shown in [14] that there exists an operator $T_0 \in \mathscr{B}(L^1, L^\infty)$ which is completely continuous on $L^1$, but fails to be so on $X$. Such an operator $T_0$ is also completely continuous on $L^p$, $1 \leqq p < \infty$ [4], [14]. Thus the necessity is proved.

If $I$ is a finite interval, then for each operator $T$ which is completely continuous on $L^\infty$, there exists a sequence of averaging operators $A_n$, $n=1, 2, \ldots$, such that $\|(I-A_n)T\|_\infty \to 0$ [5, p. 22]. This fact can be used in the proof of the following theorem.

THEOREM 5. *Let $I$ be a finite interval, and let $X$ satisfy $\mathscr{M}^p(A)$, $1 \leqq p < \infty$. In order that every $T \in \mathscr{B}(L^p, L^\infty)$ which is completely continuous on $L^\infty$ should also be completely continuous on $X$, it is necessary and sufficient that*

$$\lim_{a\to 0} a^{1/p} \|\sigma_a\|_X = 0. \tag{5.7}$$

**Proof.** The sufficiency is derived from (5.4) in a similar manner as in the proof of Theorem 4. Without loss of generality, we prove the necessity for $l=1$. First we note that the condition $\mathscr{M}^p(A)$ implies

$$a^{1/p} \|\sigma_a\|_X \leqq A \quad \textit{for all } a, 0 < a \leqq 1. \tag{5.8}$$

This follows from the relation $a^{1/p}\sigma_a f \prec^p f$, $0 < a \leqq 1$, $f \in L^p$, which can be easily verified.

Suppose that (5.7) is not true. Then there exists a $\delta > 0$ such that for arbitrarily small $a > 0$, $a^{1/p}\|\sigma_a\|_X > \delta$. For each $a$ of this kind there exists a function $g$, which we may assume positive, such that

$$g \in S; \qquad \|g\|_X \leqq 1; \qquad a^{1/p}\|\sigma_a g\|_X > \delta. \tag{5.9}$$

We can replace $g$ by $\chi_{(0,a)}g$, since this will not change $\sigma_a g$. Then, for $n \to \infty$ we will have $\chi_{(1/n,a)}g \uparrow g$, $\sigma_a(\chi_{(1/n,a)}g) \uparrow \sigma_a g$. From (1.2) it follows that we can assume that the functions $g$ in (5.9) have support $(c, a)$, $0 < c < a$. In addition to (5.9) we have

$$a^{1/2p}\|\sigma_{\surd a} g\|_X > A^{-1}\delta, \tag{5.10}$$

since by (5.8) and (5.9)

$$\delta < a^{1/p}\|\sigma_{\surd a}\|_X \|\sigma_{\surd a} g\|_X.$$

We can select a sequence of functions $g_n$, with supports $(c_n, a_n)$, $n=1, 2, \ldots$, which satisfy (5.9) and (5.10) and for which, in addition, all intervals $(c_n, a_n)$, $n=1, 2, \ldots$, are disjoint, all intervals $(c_n/(a_n)^{1/2}, (a_n)^{1/2})$ are disjoint, and $\sum a_n^{1/2p} < +\infty$.

We define the operators

$$T = \sum_{n=1}^{\infty} T_n; \qquad T_n = a_n^{1/2p}\sigma_{(a_n)^{1/2}}A_{g_n}, \qquad n = 1, 2, \ldots, \tag{5.11}$$

where $A_{g_n}$ are the averaging operators corresponding to the functions $g_n$. Then $\|T_n\|_\infty \leqq a_n^{1/2p}$; the $T_n$ are completely continuous on $L^\infty$. It follows that also $T$ is completely continuous in $L^\infty$.

For any $f$, $T_n f = T_n(f\chi_{(c_n, a_n)})$. Also, $T_n f$ has support $(c_n/(a_n)^{1/2}, (a_n)^{1/2})$. Thus, all $T_n f$ are disjoint. It is easy to see that $\|\sigma_a\|_p \leqq a^{-1/p}$, $0<a<1$. From this it follows that $\|T_n\|_p \leqq 1$. Therefore

$$\|Tf\|_p^p = \sum_{n=1}^{\infty} \|T_n f\|_p^p \leqq \sum_{n=1}^{\infty} \|f\chi_{(c_n, a_n)}\|_p^p \leqq \|f\|_p^p, \qquad f \in L^p,$$

and we see that $T \in \mathscr{B}(L^p)$.

It remains to show that $T$ is not completely continuous on $X$. For the sequence of functions $g_n$, bounded in norm in $X$, we have $Tg_n = T_n g_n$, and by (5.10), $\|T_n g_n\|_X \geqq A^{-1}\delta > 0$, also $T_n g_n(t) \to 0$ everywhere. If $Tg_n$ would have a convergent subsequence in $X$, it could converge only to 0, and this is impossible.

We turn now to the pair $(L^1, L^q)$, $1<q<\infty$. Applying similar arguments (or considering the conjugate spaces) we obtain

LEMMA 9. *If $X \in \mathscr{M}_q(A)$, $1<q<\infty$, then, for every $0 \neq T \in \mathscr{B}(L^1, L^q; K_1, K_q)$, we have*

$$\|T\|_X \leqq \lambda_p A(K_q^q K_1^{-1})^{1/(q-1)}\|\sigma_a\|_X, \tag{5.12}$$

*where $a = (K_q^q K_1^{-1})^{q/(q-1)}$.*

We also have

THEOREM 6. *Let $X \in \mathscr{M}_q(A)$, $1<q<\infty$. In order that every operator $T \in \mathscr{B}(L^1, L^q)$ which is completely continuous on $L^q$ (or $L^1$) should be also completely continuous on $X$, it is necessary and sufficient that the following condition* (5.13) (*resp.* (5.14)) *hold*:

$$\lim_{a\to 0} a\|\sigma_a\|_X = 0; \tag{5.13}$$

$$\lim_{a\to \infty} a^{1/q}\|\sigma_a\|_X = 0. \tag{5.14}$$

6. **Orlicz spaces.** In view of Examples 2 and 3 it appears to be worthwhile to give examples of classes of function spaces $X \in \mathscr{M}^p(1)$, $1 \leqq p < \infty$, for which $\|T\|_X \leqq 1$ holds for every $T \in \mathscr{B}(L^p, L^\infty; 1, 1)$.

We consider $N$-functions (compare [6]) $M$ having the expression

$$M(u) = \int_0^u (u-t)^p \, d\phi(t), \qquad u > 0, \tag{6.1}$$

where $1 \leqq p < \infty$ and $\phi$ is a positive nondecreasing left continuous function with $\phi(0)=0$. For example, for $r$ with $p \leqq r < \infty$, the $N$-function $M(u)=u^r$, $u>0$, has an expression (6.1). For an $N$-function $M$, let $L_M = L_M(I)$ denote the *Orlicz space* defined by $M$ with the norm $\|\cdot\|_M$, where,

$$\|f\|_M = \inf\{\xi : \rho_M(\xi^{-1}f) \leqq 1, \ \xi > 0\}$$

and

$$\rho_M(f) = \int_I M(|f(t)|) \, dt, \qquad f \in L_M.$$

Then we have

THEOREM 7. *Let $M$ have the expression* (6.1). *The Orlicz space $L_M$ has the interpolation property for the pair $(L^p, L^\infty)$ in the strong sense. In addition, for every $T \in \mathscr{B}(L^p, L^\infty; K_p, K_\infty)$,*

$$\|T\|_M \leqq \operatorname{Max}(K_p, K_\infty). \tag{6.2}$$

**Proof.** We may assume that $K_p = K_\infty = 1$. Let $f \in L_M$ and $T \in \mathscr{B}(L^p, L^\infty; 1, 1)$. We have

$$\rho_M(Tf) = \int_I M(|Tf(t)|) \, dt = \int_I \left\{ \int_0^{|Tf(t)|} (|Tf(t)|-s)^p \, d\phi(s) \right\} dt$$

by (6.1). By Fubini's theorem this implies

$$\rho_M(Tf) = \int_0^\infty d\phi(s) \int_{E_s} (|Tf(t)|-s)^p \, dt,$$

where $E_s$, $s>0$, is the set $\{t : |Tf(t)| > s, t \in I\}$. In view of the equality $(|Tf| - s\mathbf{1})\chi_{E_s} = |Tf| - |Tf|^{(s)} = |Tf - T(f)^{(s)}|$, the last term is equal to

$$\int_0^\infty d\phi(s) \int_I |Tf(t) - (Tf(t))^{(s)}|^p \, dt = \int_0^\infty \|Tf - (Tf)^{(s)}\|_p^p \, d\phi(s).$$

Since $T \in \mathscr{B}(L^p, L^\infty; 1, 1)$, we get

$$\|Tf - (Tf)^{(s)}\|_p^p \leqq \|Tf - T(f^{(s)})\|_p^p \leqq \|f - f^{(s)}\|_p^p,$$

which, in turn, implies $\rho_M(Tf) \leqq \rho_M(f)$. Consequently, on account of the fact that $\|f\|_M \leqq 1$ if and only if $\rho_M(f) \leqq 1$, we have $\|Tf\|_M \leqq \|f\|_M$. As $f$ is arbitrary, we obtain $\|T\|_M \leqq 1$.

In view of the proof above, we see that this theorem is also valid for Lipschitz operators acting on both $L^p$ and $L^\infty$ if the norms of the operators are now interpreted as their bounds. Thus, $\|T\|_X$ is now the smallest number $\gamma$ satisfying $\|Tf - Tg\|_X \leqq \gamma \|f-g\|_X$ for all $f, g \in X$. Since every $N$-function $M$ has the expression (6.1) for $p=1$, Theorem 6 is a generalization of a theorem by W. Orlicz [13].

## REFERENCES


1. D. W. Boyd, *The Hilbert transform on rearrangement invariant spaces*, Canad. J. Math. **19** (1967), 599–616. MR **35** #3383.

2. A. P. Calderón, *Spaces between $L^1$ and $L^\infty$ and the theorem of Marcinkiewicz*, Studia Math. **26** (1966), 273–299. MR **34** #3295.

3. G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge Univ. Press, New York, 1934.

4. M. A. Krasnosel'skiĭ, *On a theorem of M. Riesz*, Dokl. Akad. Nauk SSSR **131** (1960), 246–248 = Soviet Math. Dokl. **1** (1960), 229–231. MR **22** #9852.

5. M. A. Krasnosel'skiĭ et al., *Integral operators in spaces of summable functions*, "Nauka", Moscow, 1966. (Russian) MR **34** #6568.

6. M. A. Krasnosel'skiĭ and Ja. B. Rutickiĭ, *Convex functions and Orlicz spaces*, Noordhoff, Groningen, 1961.

7. S. G. Kreĭn and Ju. I. Petunin, *Scales of Banach spaces*, Uspehi Mat. Nauk **21** (1966), no. 2 (128), 89–168 = Russian Math. Surveys **21** (1966), no. 2, 85–159. MR **33** #1719.

8. G. G. Lorentz, *Bernstein polynomials*, Math. Expositions, no. 8, Univ. of Toronto Press, Toronto, 1953. MR **15**, 217.

9. ———, *On the theory of spaces* $\Lambda$, Pacific. J. Math. **1** (1951), 411–429. MR **13**, 470.

10. G. G. Lorentz and T. Shimogaki, *Interpolation theorems for operators in function spaces*, J. Functional Analysis **2** (1968), 31–51. MR **41** #2423.

11. ———, *Majorants for interpolation theorems*, Publ. Ramanujan Inst. No. 1 (1969), 115–122.

12. W. A. J. Luxemburg, *Rearrangement invariant Banach function spaces*, Proc. Sympos. Analysis, Queen's Univ. **10** (1967), 83–144.

13. W. Orlicz, *On a class of operations over the space of integrable functions*, Studia Math. **14** (1955), 302–309. MR **16**, 834.

14. T. Shimogaki, *On the complete continuity of operators in an interpolation theorem*, J. Fac. Sci. Hokkaido Univ. Ser. I **20** (1968), 109–114. MR **39** 2011.

15. ———, *An interpolation theorem on Banach function spaces*, Studia Math. **31** (1968), 233–240. MR **38** #2618.

16. ———, *On an equivalence relation on semi-ordered linear spaces*, J. Fac. Sci. Hokkaido Univ. Ser. I **18** (1964), 41–55. MR **30** #5149.



UNIVERSITY OF TEXAS AT AUSTIN,
AUSTIN, TEXAS 78712
TOKYO INSTITUTE OF TECHNOLOGY,
TOKYO, JAPAN